Strong-property-fluctuation theory for homogenization of bianisotropic composites: Formulation

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The strong-property-fluctuation theory is developed for the homogenization of the linear dielectric, magnetic, and magnetoelectric properties of a two-constituent bianisotropic composite. The notion of a bianisotropic comparison medium (BCM) is introduced to serve as a springboard for the Dyson equation satisfied by the ensemble-averaged electromagnetic field. With the constitutive properties of the BCM serving as the zeroth-order solution of the Dyson equation, the first-order correction, known as the bilocal approximation, is obtained. Wave propagation in the composite can be described in this manner by a nonlocal effective medium containing information about the spatial correlations of the constitutive properties. For scales larger than the correlation length, the nonlocality vanishes and a local effective medium emerges. Analytical results for the local effective constitutive properties are presented after assuming a spherical particulate topology for the constituent mediums. Illustrative numerical results are provided.

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I. INTRODUCTION

The fabrication of composite materials provides an effective way for combining the desirable electromagnetic properties of two or more different materials, provided the constituent material phases do not chemically react with each other. Prediction of the effective electromagnetic properties of linear composites, from the properties of their constituents, has been a focus of research for over two centuries and it continues to be a matter of considerable scientific and technological importance [1,2]. Until recently, however, analyses have been confined to isotropic dielectric and/or magnetic composites. Furthermore, the limitations of most homogenization approaches—exemplified by the Maxwell Garnett and the Bruggeman formalisms, and their variants—arise from their simplistic treatments of the distributional statistics of the constituent phases.

A notable exception is the so-called strong-property-fluctuation theory (SPFT), which provides a method to determine both local and nonlocal constitutive properties of composites while allowing for a sophisticated handling of the distributional statistics [3]. In the SPFT a preliminary ansatz is made about the nature of the composite; the ansatz is used to perturbatively calculate corrections in orders of statistical cumulants of the spatial distribution of the constituent phases. The appeal of SPFT lies in its generality: even the simplest SPFT result represents an advancement over the Bruggeman formalism. The theory has already been developed for isotropic dielectric [3], anisotropic dielectric [4], as well as chiral-in-chiral composites [5]. Additionally, the SPFT is not restricted only to particulate composites.

During the past 15 years, there has been an explosion in the literature on bianisotropic materials, both on theoretical and experimental aspects [6–9]. Bianisotropic materials are characterized by three types of constitutive properties: dielectric, magnetic, and magnetoelectric. Composite bianisotropic materials have been treated by the Maxwell Garnett and the Bruggeman formalisms [10], but not yet by the SPFT. The chief difficulty in the application of SPFT arises from the source-region singularity of the corresponding dyadic Green function which can result in the generation of *secular terms* (i.e., terms resulting in divergence) in the perturbation expansion of the electromagnetic-field equations [11,12]. However, Michel and Weiglhofer [13] recently developed a treatment of this singularity in bianisotropic mediums, thereby enabling the SPFT formulation for bianisotropic-in-bianisotropic composites. Accordingly, we are initiating a research program in this direction, this paper being the first of a series.

The objectives of the present study are twofold. The first is to generalize the SPFT to bianisotropic composites. In so doing we follow closely the argumentation of Michel and Lakhtakia [5] for isotropic chiral composites. Allowance is made for a nonisotropic distribution of constituent material phases, such as may arise if the constituent phases comprise ellipsoidal particles. The second objective is to implement the developed equations in the case of reciprocal biaxial bianisotropic composites.

The layout of this paper is as follows: Following a general presentation of the statistical parameters used to describe bianisotropic composites, we introduce the notion of a *bianisotropic comparison medium* (BCM). The BCM is a local homogeneous medium which plays a central role in SPFT. We

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develop the *Dyson equation* for the average electromagnetic field in a two-phase composite and utilize the BCM in a lowest-order estimate of the effective medium electromagnetic properties. This lowest-order estimate—which is equivalent to that provided by the Bruggeman formalism for bianisotropic composites—serves as our initial ansatz in the iterative process resulting in the SPFT estimate of constitutive properties. The first-order correction, known as the *bilocal approximation*, is then presented. Next we derive the relation between the constitutive dyadics of the exact effective medium (which includes all correlation effects) and the so-called *mass operator* of the Dyson equation. We then consider the case where principal electromagnetic wavelengths are long compared with the correlation length, and the composite medium can therefore be regarded as homogeneous. Finally we implement the SPFT, in the long-wavelength and bilocal approximations, for the case of reciprocal bianisotropic composites. Both chiral and biaxial composites are considered in the illustrative numerical results presented here, and a detailed numerical study is planned to appear later [14]. The SPFT estimates of the constitutive properties of all composites considered are compared and contrasted with those provided by the Bruggeman and the incremental Maxwell Garnett formalisms [15,16].

We adopt the following notation: three-vectors (six-vectors) are in normal (bold) face and underlined, whereas 3×3 dyadics (6×6 dyadics) are in normal (bold) face and underlined twice. The adjoint, determinant, inverse, and trace of the dyadic $\underline{\underline{\mathbf{Q}}}$ are denoted by adj$(\underline{\underline{\mathbf{Q}}})$, det $\underline{\underline{\mathbf{Q}}}$, $\underline{\underline{\mathbf{Q}}}^{-1}$, and tr $\underline{\underline{\mathbf{Q}}}$, respectively.

II. GENERAL

We start with the frequency-dependent version of the source-free Maxwell curl postulates,

$$\underline{\nabla}\times\underline{E}(\underline{r}) = i\omega\underline{B}(\underline{r}), \tag{1}$$

$$\underline{\nabla}\times\underline{H}(\underline{r}) = -i\omega\underline{D}(\underline{r}), \tag{2}$$

where we have assumed an $\exp(-i\omega t)$ time dependence with ω as the angular frequency. The constitutive relations of a nonhomogeneous bianisotropic medium are given as

$$\underline{D}(\underline{r}) = \underline{\underline{\epsilon}}(\underline{r})\cdot\underline{E}(\underline{r}) + \underline{\underline{\xi}}(\underline{r})\cdot\underline{H}(\underline{r}), \tag{3}$$

$$\underline{B}(\underline{r}) = \underline{\underline{\zeta}}(\underline{r})\cdot\underline{E}(\underline{r}) + \underline{\underline{\mu}}(\underline{r})\cdot\underline{H}(\underline{r}), \tag{4}$$

where $\underline{\underline{\epsilon}}(\underline{r})$ and $\underline{\underline{\mu}}(\underline{r})$ are the permittivity and permeability dyadics, respectively, and $\underline{\underline{\xi}}(\underline{r})$ and $\underline{\underline{\zeta}}(\underline{r})$ are the magnetoelectric dyadics. Equations (1)–(4) can be represented compactly in six-vector/dyadic notation as

$$\underline{\underline{\mathbf{L}}}(\nabla)\cdot\underline{\mathbf{F}}(\underline{r}) = -i\omega\underline{\underline{\mathbf{K}}}(\underline{r})\cdot\underline{\mathbf{F}}(\underline{r}), \tag{5}$$

$$\underline{\mathbf{C}}(\underline{r}) = \underline{\underline{\mathbf{K}}}(\underline{r})\cdot\underline{\mathbf{F}}(\underline{r}), \tag{6}$$

where

$$\underline{\underline{\mathbf{L}}}(\nabla) = \begin{bmatrix} \underline{\underline{0}} & \nabla\times\underline{\underline{I}} \\ -\nabla\times\underline{\underline{I}} & \underline{\underline{0}} \end{bmatrix}, \quad \underline{\underline{\mathbf{K}}}(\underline{r}) = \begin{bmatrix} \underline{\underline{\epsilon}}(\underline{r}) & \underline{\underline{\xi}}(\underline{r}), \\ \underline{\underline{\zeta}}(\underline{r}) & \underline{\underline{\mu}}(\underline{r}) \end{bmatrix}, \tag{7}$$

and

$$\underline{\mathbf{F}}(\underline{r}) = \begin{bmatrix} \underline{E}(\underline{r}) \\ \underline{H}(\underline{r}) \end{bmatrix}, \quad \underline{\mathbf{C}}(\underline{r}) = \begin{bmatrix} \underline{D}(\underline{r}) \\ \underline{B}(\underline{r}) \end{bmatrix}, \tag{8}$$

with $\underline{\underline{I}}$ denoting the 3×3 unit dyadic.

We consider a two-phase composite consisting of two bianisotropic constituent phases mixed at the microscopic, but not molecular, length scale. Let all space be divided into disjoint parts V_a and V_b containing the phases labeled a and b, respectively. For $\underline{r}\in V_p$ $(p=a,b)$, we write

$$\underline{\underline{\epsilon}}(\underline{r}) = \underline{\underline{\epsilon}}_p, \quad \underline{\underline{\xi}}(\underline{r}) = \underline{\underline{\xi}}_p, \quad \underline{\underline{\zeta}}(\underline{r}) = \underline{\underline{\zeta}}_p, \quad \underline{\underline{\mu}}(\underline{r}) = \underline{\underline{\mu}}_p, \quad \underline{r}\in V_p, \tag{9}$$

so that

$$\underline{\underline{\mathbf{K}}}(\underline{r}) = \underline{\underline{\mathbf{K}}}_p, \quad \underline{r}\in V_p. \tag{10}$$

We introduce two characteristic functions θ_p as

$$\theta_p(\underline{r}) = \begin{cases} 1, & \underline{r}\in V_p \\ 0, & \underline{r}\notin V_p, \end{cases} \tag{11}$$

thus

$$\theta_a(\underline{r}) + \theta_b(\underline{r}) = 1, \quad \underline{r}\in V_a\cup V_b. \tag{12}$$

Any of the $\underline{r}$-dependent constitutive quantities can be expressed *everywhere* in terms of the characteristic functions $\theta_p(\underline{r})$; for example,

$$\underline{\underline{\mathbf{K}}}(\underline{r}) = \underline{\underline{\mathbf{K}}}_a\theta_a(\underline{r}) + \underline{\underline{\mathbf{K}}}_b\theta_b(\underline{r}), \quad \underline{r}\in V_a\cup V_b. \tag{13}$$

Throughout this work we use the concept of ensemble averaging, i.e., averaging over a large number of different samples of the two-phase composite, and we denote ensemble averages by $\langle\,\rangle$. The complete statistical information about the composite is contained in *moments* of the characteristic function $\theta_a(\underline{r})$. The nth moment is the expectation value $\langle\theta_a(\underline{r}_1)\cdots\theta_a(\underline{r}_n)\rangle$ and represents the probability for $\underline{r}_1,\ldots,\underline{r}_n$ being inside V_a; equivalently, we may use b instead of a due to Eq. (12). We assume that, on average, the composite is homogeneous.

The first moment for the phase a is its volume fraction

$$f_a = \langle\theta_a(\underline{r})\rangle, \tag{14}$$

which is constant with respect to $\underline{r}$. The same holds for the volume fraction $f_b = \langle\theta_b(\underline{r})\rangle$ of phase b. Obviously, $f_a + f_b = 1$.

The two volume fractions f_a and f_b contain only minimal geometrical information about the composite. A more detailed description is provided by the second moment $\langle\theta_a(\underline{r})\theta_a(\underline{r}')\rangle$ of $\theta_a(\underline{r})$, or, equivalently, by the second *cumulant* or covariance

$$\tau(\underline{R}) = \langle \theta_a(\underline{r})\theta_a(\underline{r}')\rangle - \langle \theta_a(\underline{r})\rangle\langle \theta_a(\underline{r}')\rangle = \langle \theta_b(\underline{r})\theta_b(\underline{r}')\rangle - \langle \theta_b(\underline{r})\rangle\langle \theta_b(\underline{r}')\rangle, \tag{15}$$

where $\underline{R} = \underline{r} - \underline{r}'$. If the composite is disordered, it is usually possible to define a correlation length L such that $\tau(\underline{R})$ is negligible for $|\underline{R}| \gg L$, i.e., on scales larger than L, the composite may be considered homogeneous.

The formulation of SPFT requires the introduction of a bianisotropic comparison medium, which allows an approximate treatment of electromagnetic fields in $V_a \cup V_b$. The constitutive dyadics $\underline{\underline{\epsilon}}_{\mathrm{BCM}}$, $\underline{\underline{\xi}}_{\mathrm{BCM}}$, $\underline{\underline{\zeta}}_{\mathrm{BCM}}$, and $\underline{\underline{\mu}}_{\mathrm{BCM}}$ of this medium are not $\underline{r}$ dependent; hence it is not only homogeneous but also spatially *local*. The BCM will later on serve as the preliminary ansatz for the SPFT and will be shown in Sec. III to actually be in agreement with the Bruggeman formalism. Electromagnetic wave propagation in the BCM is described by

$$\underline{\underline{\mathbf{L}}}(\nabla)\cdot \underline{\mathbf{F}}_{BCM}(\underline{r}) = -i\omega \underline{\underline{\mathbf{K}}}_{BCM}\cdot \underline{\mathbf{F}}_{BCM}(\underline{r}), \tag{16}$$

where

$$\underline{\underline{\mathbf{K}}}_{BCM} = \begin{bmatrix} \underline{\underline{\epsilon}}_{BCM} & \underline{\underline{\xi}}_{BCM} \\ \underline{\underline{\zeta}}_{BCM} & \underline{\underline{\mu}}_{BCM} \end{bmatrix}, \quad \underline{\mathbf{F}}_{BCM}(\underline{r}) = \begin{bmatrix} \underline{E}_{BCM}(\underline{r}) \\ \underline{H}_{BCM}(\underline{r}) \end{bmatrix}, \tag{17}$$

with $\underline{\mathbf{F}}_{BCM}(\underline{r})$ denoting the local spatially averaged electromagnetic field. We introduce the 6×6 dyadic Green function $\underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}-\underline{r}')$ which satisfies the differential equation

$$[\underline{\underline{\mathbf{L}}}(\nabla) + i\omega \underline{\underline{\mathbf{K}}}_{BCM}]\cdot \underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}-\underline{r}') = \underline{\underline{\mathbf{I}}}\delta(\underline{r}-\underline{r}'), \tag{18}$$

where $\underline{\underline{\mathbf{I}}}$ is the unit 6×6 dyadic and $\delta(\underline{r}-\underline{r}')$ is the Dirac δ function. The singular behavior of $\underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}-\underline{r}')$ in the limit $\underline{r}\to\underline{r}'$ can be accommodated through

$$\underline{\underline{\mathbf{G}}}_{BCM}(\underline{\mathrm{R}}) = \mathrm{P}\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R}) + \underline{\underline{\mathbf{D}}}\delta(\underline{R}), \tag{19}$$

where P is the principal value operation excluding a certain infinitesimal region centered on $\underline{R} = \underline{0}$ and $\underline{\underline{\mathbf{D}}}$ is the corresponding depolarization dyadic of the specified region in the BCM [13]. The dyadic $\underline{\underline{\mathbf{D}}}$ is fixed at a later stage in the analysis.

III. DYSON EQUATION

With the foregoing generalities established, we now proceed to derive the central equation in the SPFT: the Dyson equation. From Eq. (5) we obtain

$$[\underline{\underline{\mathbf{L}}}(\nabla) + i\omega \underline{\underline{\mathbf{K}}}_{BCM}]\cdot \underline{\mathbf{F}}(\underline{r}) = -i\omega[\underline{\underline{\mathbf{K}}}(\underline{r}) - \underline{\underline{\mathbf{K}}}_{BCM}]\cdot \underline{\mathbf{F}}(\underline{r}). \tag{20}$$

By virtue of Eqs. (16) and (18), the solution of Eq. (20) may be represented by the following *Fredholm equation of the third kind* [17]:

$$\underline{\mathbf{F}}(\underline{r}) = \underline{\mathbf{F}}_{BCM}(\underline{r}) - i\omega\int \underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}-\underline{r}')\cdot[\underline{\underline{\mathbf{K}}}(\underline{r}') - \underline{\underline{\mathbf{K}}}_{BCM}]\cdot \underline{\mathbf{F}}(\underline{r}')d^3\underline{r}'. \tag{21}$$

Clearly, $\underline{\mathbf{F}}_{BCM}(\underline{r})$ now serves as a solution of the homogeneous version of (20), i.e., $\underline{\mathbf{F}}_{BCM}(\underline{r})$ is the *complementary function*. Here and hereafter, integration is performed within infinite limits if the domain of integration is not indicated explicitly.

Equation (21) cannot be evaluated perturbatively when the constitutive parameters in $\underline{\underline{\mathbf{K}}}(\underline{r})$ fluctuate strongly. This is due to secular terms produced by the singularities of the dyadic Green function $\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R})$ in the source region. The singularities can be removed from the right side of Eq. (21) by taking advantage of Eq. (19); thus,

$$\underline{\mathbf{F}}(\underline{r}) = \underline{\mathbf{F}}_{BCM}(\underline{r}) - i\omega\mathrm{P}\int \underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}-\underline{r}')\cdot[\underline{\underline{\mathbf{K}}}(\underline{r}') - \underline{\underline{\mathbf{K}}}_{BCM}]\cdot \underline{\mathbf{F}}(\underline{r}')d^3\underline{r}' - i\omega\underline{\underline{\mathbf{D}}}\cdot[\underline{\underline{\mathbf{K}}}(\underline{r}) - \underline{\underline{\mathbf{K}}}_{BCM}]\cdot \underline{\mathbf{F}}(\underline{r}). \tag{22}$$

Next, after introducing the *exciting field*

$$\underline{\mathbf{F}}_{exc}(\underline{r}) = \{\underline{\underline{\mathbf{I}}} + i\omega\underline{\underline{\mathbf{D}}}\cdot[\underline{\underline{\mathbf{K}}}(\underline{r}) - \underline{\underline{K}}_{BCM}]\}\cdot \underline{\mathbf{F}}(\underline{r}), \tag{23}$$

we rewrite the integral equation (22) as

$$\underline{\mathbf{F}}_{exc}(\underline{r}) = \underline{\mathbf{F}}_{BCM}(\underline{r}) + \mathrm{P}\int \underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}-\underline{r}')\cdot \underline{\underline{\boldsymbol{\chi}}} \times(\underline{r}')\cdot \underline{\mathbf{F}}_{exc}(\underline{r}')\cdot \underline{\mathbf{F}}_{exc}(\underline{r}')d^3\underline{r}', \tag{24}$$

with a generalized *polarizability dyadic* defined as

$$\underline{\underline{\boldsymbol{\chi}}}(\underline{r}) = -i\omega[\underline{\underline{\mathbf{K}}}(\underline{r}) - \underline{\underline{\mathbf{K}}}_{BCM}]\cdot\{\underline{\underline{\mathbf{I}}} + i\omega\underline{\underline{\mathbf{D}}}\cdot[\underline{\underline{\mathbf{K}}}(\underline{r}) - \underline{\underline{\mathbf{K}}}_{BCM}]\}^{-1}. \tag{25}$$

The next steps are canonical: we calculate the ensemble average $\langle \underline{\mathbf{F}}_{exc}(\underline{r})\rangle$ of the exciting field by ensemble averaging both sides of the integral equation (24). For this purpose, we formally represent the equation in terms of a Born series and average each term of the series separately [12]. We fix the lowest-order estimate of the effective-medium properties by demanding that

$$\langle \underline{\underline{\boldsymbol{\chi}}}(\underline{r})\rangle = \underline{\underline{\mathbf{0}}}, \tag{26}$$

which condition removes the secular terms from the Born series expansion [11]. Inserting Eqs. (13) and (14) into Eq. (26), we obtain

$$(\underline{\underline{\mathbf{K}}}_a - \underline{\underline{\mathbf{K}}}_{BCM})\cdot[\underline{\underline{\mathbf{I}}} + i\omega\underline{\underline{\mathbf{D}}}\cdot(\underline{\underline{\mathbf{K}}}_a - \underline{\underline{\mathbf{K}}}_{BCM})]^{-1}f_a + (\underline{\underline{\mathbf{K}}}_b - \underline{\underline{\mathbf{K}}}_{BCM})\cdot[\underline{\underline{\mathbf{I}}} + i\omega\underline{\underline{\mathbf{D}}}\cdot(\underline{\underline{\mathbf{K}}}_b - \underline{\underline{\mathbf{K}}}_{BCM})]^{-1}f_b = \underline{\underline{\mathbf{0}}}, \tag{27}$$

which is the Bruggeman equation for bianisotropic composites [18]. Thus, we see that electromagnetic wave propagation in $V_a \cup V_b$ can indeed be approximately described by means of the BCM.

Equation (24) may now be ensemble averaged using the Feynman-diagrammatic technique introduced by Frisch [12] to arrive at the Dyson equation

$$\langle \underline{\mathbf{F}}_{exc}(\underline{r}) \rangle = \underline{\mathbf{F}}_{BCM}(\underline{r}) + \mathrm{P}\int \underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}-\underline{r}') \cdot \left\{ \int \underline{\underline{\boldsymbol{\Sigma}}}(\underline{r}'-\underline{r}'') \cdot \langle \underline{\mathbf{F}}_{exc}(\underline{r}'') \rangle d^3\underline{r}'' \right\} d^3\underline{r}', \tag{28}$$

where the quantity $\underline{\underline{\boldsymbol{\Sigma}}}(\underline{r}'-\underline{r}'')$ is called the *mass operator*. The mass operator consists of an infinite series, each term of which contains products over $\mathrm{P}\underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}'-\underline{r}'')$ and the statistical cumulants of $\underline{\underline{\boldsymbol{\chi}}}(\underline{r}')$. In practice, approximations to the Dyson equation are unavoidable. They are usually implemented by truncating the series expansion of the mass operator $\underline{\underline{\boldsymbol{\Sigma}}}$. To the lowest (i.e., second) order in $\underline{\underline{\boldsymbol{\chi}}}$ we have

$$\underline{\underline{\boldsymbol{\Sigma}}}(\underline{r}-\underline{r}') = \langle \underline{\underline{\boldsymbol{\chi}}}(\underline{r}) \cdot \mathrm{P}\underline{\underline{\mathbf{G}}}_{BCM}(\underline{r}-\underline{r}') \cdot \underline{\underline{\boldsymbol{\chi}}}(\underline{r}') \rangle, \tag{29}$$

which is called the *bilocal approximation* [19]. Since

$$\underline{\underline{\boldsymbol{\chi}}}(\underline{r}) = \underline{\underline{\boldsymbol{\chi}}}_a \theta_a(\underline{r}) + \underline{\underline{\boldsymbol{\chi}}}_b \theta_b(\underline{r}), \tag{30}$$

Eq. (29) leads to

$$\underline{\underline{\boldsymbol{\Sigma}}}(\underline{R}) = \tau(\underline{R})(\underline{\underline{\boldsymbol{\chi}}}_a - \underline{\underline{\boldsymbol{\chi}}}_b) \cdot \mathrm{P}\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R}) \cdot (\underline{\underline{\boldsymbol{\chi}}}_a - \underline{\underline{\boldsymbol{\chi}}}_b) \tag{31}$$

after some algebraic manipulations exploiting Eq. (26), the covariance $\tau(\underline{R})$ having been introduced in Eq. (15).

IV. NONLOCAL EFFECTIVE MEDIUM

In order to complete the SPFT formulation, we go on to determine the relation between the ensemble-averaged fields $\langle \underline{\mathbf{C}}(\underline{r}) \rangle$ and $\langle \underline{\mathbf{F}}(\underline{r}) \rangle$. The ensemble average of the constitutive relation (6) may be stated as

$$\langle \underline{\mathbf{C}}(\underline{r}) \rangle = \langle \underline{\underline{\mathbf{K}}}(\underline{r}) \cdot \underline{\mathbf{F}}(\underline{r}) \rangle. \tag{32}$$

The relationship between $\langle \underline{\underline{\mathbf{K}}}(\underline{r}) \cdot \underline{\mathbf{F}}(\underline{r}) \rangle$ and $\langle \underline{\mathbf{F}}(\underline{r}) \rangle$ must be linear, because the composite is linear. Furthermore, this relation has to be of the form of a convolution integral

$$\langle \underline{\underline{\mathbf{K}}}(\underline{r}) \cdot \underline{\mathbf{F}}(\underline{r}) \rangle = \int \underline{\underline{\mathbf{K}}}_{Dy}(\underline{R}) \cdot \langle \underline{\mathbf{F}}(\underline{r}-\underline{R}) \rangle d^3\underline{R} \tag{33}$$

due to translational invariance. The dyadic $\underline{\underline{\mathbf{K}}}_{Dy}(\underline{R})$ contains the constitutive properties of the effective medium consistent with the SPFT. In general, $\underline{\underline{\mathbf{K}}}_{Dy}(\underline{R})$ is *spatially nonlocal* and, therefore, signifies spatial dispersion.

Equations (23) and (25) yield

$$\underline{\underline{\boldsymbol{\chi}}}(\underline{r}) \cdot \underline{\mathbf{F}}_{exc}(\underline{r}) = -i\omega[\underline{\underline{\mathbf{K}}}(\underline{r}) - \underline{\underline{\mathbf{K}}}_{BCM}] \cdot \underline{\mathbf{F}}(\underline{r}), \tag{34}$$

whence

$$\langle \underline{\underline{\boldsymbol{\chi}}}(\underline{r}) \cdot \underline{\mathbf{F}}_{exc}(\underline{r}) \rangle = -i\omega[\langle \underline{\underline{\mathbf{K}}}(\underline{r}) \cdot \underline{\mathbf{F}}(\underline{r}) \rangle - \underline{\underline{\mathbf{K}}}_{BCM} \cdot \langle \underline{\mathbf{F}}(\underline{r}) \rangle]. \tag{35}$$

The ensemble-averaged counterpart of Eq. (23) is given by

$$\langle \underline{\mathbf{F}}_{exc}(\underline{r}) \rangle = (\underline{\underline{\mathbf{I}}} - i\omega \underline{\underline{\mathbf{D}}} \cdot \underline{\underline{\mathbf{K}}}_{BCM}) \cdot \langle \underline{\mathbf{F}}(\underline{r}) \rangle + i\omega \underline{\underline{\mathbf{D}}} \cdot \langle \underline{\underline{\mathbf{K}}}(\underline{r}) \cdot \underline{\mathbf{F}}(\underline{r}) \rangle. \tag{36}$$

Furthermore, on taking the ensemble average of Eq. (24) and comparing it with the Dyson equation (28), we get

$$\langle \underline{\underline{\boldsymbol{\chi}}}(\underline{r}) \cdot \underline{\mathbf{F}}_{exc}(\underline{r}) \rangle = \int \underline{\underline{\boldsymbol{\Sigma}}}(\underline{r}-\underline{r}') \cdot \langle \underline{\mathbf{F}}_{exc}(\underline{r}') \rangle d^3\underline{r}'. \tag{37}$$

Finally, after rearranging Eqs. (35)–(37) and inserting the ensemble-averaged constitutive relation (32) in Eq. (33), we obtain

$$\langle \underline{\mathbf{C}}(\underline{r}) \rangle + \int \underline{\underline{\boldsymbol{\Sigma}}}(\underline{r}-\underline{r}') \cdot \underline{\underline{\mathbf{D}}} \cdot \langle \underline{\mathbf{C}}(\underline{r}') \rangle d^3\underline{r}' = \underline{\underline{\mathbf{K}}}_{BCM} \cdot \langle \underline{\mathbf{F}}(\underline{r}) \rangle - \frac{1}{i\omega} \int \underline{\underline{\boldsymbol{\Sigma}}}(\underline{r}-\underline{r}') \cdot (\underline{\underline{\mathbf{I}}} - i\omega \underline{\underline{\mathbf{D}}} \cdot \underline{\underline{\mathbf{K}}}_{BCM}) \cdot \langle \underline{\mathbf{F}}(\underline{r}') \rangle d^3\underline{r}'. \tag{38}$$

This integral equation gives a linear relation between $\langle \underline{\mathbf{C}}(\underline{r}) \rangle$ and $\langle \underline{\mathbf{F}}(\underline{r}) \rangle$. Its solution for $\langle \underline{\mathbf{C}}(\underline{r}) \rangle$ enables the emergence of the desired constitutive dyadic $\underline{\underline{\mathbf{K}}}_{Dy}(\underline{R})$ of the nonlocal effective medium.

Since the integral equation (38) is of the convolution type, it can be solved by the Fourier-transform technique [20]. Therefore, we define the following quantities:

$$\underline{\tilde{\mathbf{F}}}(\underline{q}) = \int \langle \underline{\mathbf{F}}(\underline{r}) \rangle \exp(-i\underline{q} \cdot \underline{r}) d^3\underline{r}, \tag{39}$$

$$\underline{\tilde{\mathbf{C}}}(\underline{q}) = \int \langle \underline{\mathbf{C}}(\underline{r}) \rangle \exp(-i\underline{q} \cdot \underline{r}) d^3\underline{r}, \tag{40}$$

$$\tilde{\underline{\underline{\boldsymbol{\Sigma}}}}(\underline{q}) = \int \underline{\underline{\boldsymbol{\Sigma}}}(\underline{r}) \exp(-i\underline{q}\cdot\underline{r})\, d^3\underline{r}, \tag{41}$$

$$\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{q}) = \int \underline{\underline{\mathbf{K}}}_{Dy}(\underline{r}) \exp(-i\underline{q}\cdot\underline{r})\, d^3\underline{r}, \tag{42}$$

$\underline{q}$ being the three-dimensional spatial frequency vector. The Fourier-transformed version of Eq. (38) reads as follows:

$$[\underline{\underline{\mathbf{I}}} + \tilde{\underline{\underline{\boldsymbol{\Sigma}}}}(\underline{q})\cdot\underline{\underline{D}}]\cdot\tilde{\underline{\mathbf{C}}}(\underline{q}) = \left[\underline{\underline{\mathbf{K}}}_{BCM} - \frac{1}{i\omega}\tilde{\underline{\underline{\boldsymbol{\Sigma}}}}(\underline{q})\cdot(\underline{\underline{\mathbf{I}}} - i\omega\underline{\underline{\mathbf{D}}}\cdot\underline{\underline{\mathbf{K}}}_{BCM})\right]\cdot\tilde{\underline{\mathbf{F}}}(\underline{q}). \tag{43}$$

But $\tilde{\underline{\mathbf{C}}}(\underline{q}) = \tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{q})\cdot\tilde{\underline{\mathbf{F}}}(\underline{q})$ by virtue of the foregoing relations; hence, Eq. (43) yields

$$\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{q}) = \underline{\underline{\mathbf{K}}}_{BCM} - \frac{1}{i\omega}[\underline{\underline{\mathbf{I}}} + \tilde{\underline{\underline{\boldsymbol{\Sigma}}}}(\underline{q})\cdot\underline{\underline{\mathbf{D}}}]^{-1}\cdot\tilde{\underline{\underline{\Sigma}}}(\underline{q}). \tag{44}$$

The constitutive dyadic $\underline{\underline{\mathbf{K}}}_{Dy}(\underline{r})$ then emerges as the inverse Fourier integral

$$\underline{\underline{\mathbf{K}}}_{Dy}(\underline{r}) = \frac{1}{(2\pi)^3}\int \tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{q}) \exp(i\underline{q}\cdot\underline{r})\, d^3\underline{q}. \tag{45}$$

The Dyson equation (28) involves the ensemble-averaged exciting field $\langle\underline{\mathbf{F}}_{exc}(\underline{r})\rangle$. In order to determine the ensemble-averaged electromagnetic field $\langle\underline{\mathbf{F}}(\underline{r})\rangle$ itself, we take the ensemble average of Eq. (5) and use Eq. (33) to get

$$\underline{\underline{\mathbf{L}}}(\nabla)\cdot\langle\underline{\mathbf{F}}(\underline{r})\rangle + i\omega\int \underline{\underline{\mathbf{K}}}_{Dy}(\underline{R})\cdot\langle\underline{\mathbf{F}}(\underline{r}-\underline{R})\rangle\, d^3\underline{R} = \underline{\mathbf{0}}. \tag{46}$$

The Fourier-transformed version of this equation is

$$\left[\begin{pmatrix} \underline{\underline{0}} & i\underline{q}\times\underline{\underline{\mathbf{I}}} \\ -i\underline{q}\times\underline{\underline{\mathbf{I}}} & \underline{\underline{0}} \end{pmatrix} + i\omega\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{q})\right]\cdot\tilde{\underline{\mathbf{F}}}(\underline{q}) = \underline{\mathbf{0}}, \tag{47}$$

from which $\tilde{\underline{\mathbf{F}}}(\underline{q})$ may be extracted by standard dyadic techniques [21]. Thus, depending on a specific choice for the evaluation of $\underline{\underline{\mathbf{D}}}$, *the SPFT homogenization formulation is now complete in the bilocal approximation*.

V. LOCAL EFFECTIVE MEDIUM

When the principal electromagnetic wavelengths are much larger than the correlation length L, we can achieve a *macroscopic* description of the composite as a homogeneous local continuum [1]. Although it has a different provenance, this description is conceptually no different from that available from the Maxwell Garnett and the Bruggeman formalisms: the mixture is considered homogeneous in the so-called *long-wavelength approximation*.

Suppose the long-wavelength approximation is appropriate. Let us then introduce the macroscopic fields $\underline{\mathbf{C}}_{macro}(\underline{r})$ and $\underline{\mathbf{F}}_{macro}(\underline{r})$ by *spatially* averaging the microscopic fields $\langle\underline{\mathbf{C}}(\underline{r})\rangle$ and $\langle\underline{\mathbf{F}}(\underline{r})\rangle$ over a region V; thus,

$$\underline{\mathbf{C}}_{macro}(\underline{r}) = \frac{1}{V}\int_V \langle\underline{\mathbf{C}}(\underline{r}+\underline{r}'')\rangle\, d^3\underline{r}'', \tag{48}$$

$$\underline{\mathbf{F}}_{macro}(\underline{r}) = \frac{1}{V}\int_V \langle\underline{\mathbf{F}}(\underline{r}+\underline{r}'')\rangle\, d^3\underline{r}''. \tag{49}$$

The minimum linear cross-sectional extent of the region V must be larger than L, but smaller than the maximum principal electromagnetic wavelength. Inserting Eqs. (32) and (33) into Eq. (48), we find

$$\begin{aligned}\underline{\mathbf{C}}_{macro}(\underline{r}) &= \frac{1}{V}\int_V\left(\int \underline{\underline{\mathbf{K}}}_{Dy}(\underline{R})\cdot\langle\underline{\mathbf{F}}(\underline{r}+\underline{r}''-\underline{R})\rangle\, d^3\underline{R}\right) d^3\underline{r}'' \\ &= \int \underline{\underline{\mathbf{K}}}_{Dy}(\underline{R})\cdot\underline{\mathbf{F}}_{macro}(\underline{r}-\underline{R})\, d^3\underline{R} \\ &\approx \int \underline{\underline{\mathbf{K}}}_{Dy}(\underline{R})\cdot\underline{\mathbf{F}}_{macro}(\underline{r})\, d^3\underline{R}.\end{aligned} \tag{50}$$

This leads to the macroscopic constitutive relation

$$\underline{\mathbf{C}}_{macro}(\underline{r}) = \tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{0})\cdot\underline{\mathbf{F}}_{macro}(\underline{r}). \tag{51}$$

The constitutive properties of a two-phase bianisotropic composite in the long-wavelength approximation are thus specified by the dyadic $\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{0})$. Evidently from Eqs. (44), (41), and (31), the key step in estimating $\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{0})$ is the evaluation of

$$\begin{aligned}\tilde{\underline{\underline{\boldsymbol{\Sigma}}}}(\underline{0}) &= \int \underline{\underline{\boldsymbol{\Sigma}}}(\underline{R})\, d^3\underline{R} \\ &= (\underline{\underline{\boldsymbol{\chi}}}_a - \underline{\underline{\boldsymbol{\chi}}}_b)\cdot\left[\mathrm{P}\int \tau(\underline{R})\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R})\, d^3\underline{R}\right]\cdot(\underline{\underline{\boldsymbol{\chi}}}_a - \underline{\underline{\boldsymbol{\chi}}}_b).\end{aligned} \tag{52}$$

We note that the presence of $\tau(\underline{R})$ within the above principal-value integration is justified, because $\tau(\underline{0}) = f_a(1-f_a) = f_b(1-f_b)$ cannot be null-valued for nontrivial problems.

Although an explicit expression for $\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R})$ cannot be written down, its Fourier transform

$$\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q}) = \int \underline{\underline{\mathbf{G}}}_{BCM}(\underline{R}) \exp(-i\underline{q}\cdot\underline{R})\, d^3\underline{R} \tag{53}$$

can be obtained by taking the Fourier transforms of both sides of Eq. (18). Thus,

$$\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q}) = \frac{1}{i\omega}\frac{\mathrm{adj}[\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{q})]}{\det\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{q})}, \tag{54}$$

where

$$\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{q}) = \begin{bmatrix} \underline{\underline{0}} & (\underline{q}/\omega)\times\underline{\underline{\mathbf{I}}} \\ -(\underline{q}/\omega)\times\underline{\underline{\mathbf{I}}} & \underline{\underline{0}} \end{bmatrix} + \underline{\underline{\mathbf{K}}}_{BCM}. \tag{55}$$

For later convenience, we note that Eq. (55) can be manipulated to deliver $\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q})$ in the following general form:

$$\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q})=\frac{1}{i\omega}\left(\frac{\underline{\underline{\mathbf{T}}}_4(\hat{\underline{q}})(q/\omega)^4+\underline{\underline{\mathbf{T}}}_3(\hat{\underline{q}})(q/\omega)^3+\underline{\underline{\mathbf{T}}}_2(\hat{\underline{q}})(q/\omega)^2+\underline{\underline{\mathbf{T}}}_1(\hat{\underline{q}})(q/\omega)+\underline{\underline{\mathbf{T}}}_0(\hat{\underline{q}})}{t_4(\hat{\underline{q}})(q/\omega)^4+t_3(\hat{\underline{q}})(q/\omega)^3+t_2(\hat{\underline{q}})(q/\omega)^2+t_1(\hat{\underline{q}})(q/\omega)+t_0(\hat{\underline{q}})}\right). \tag{56}$$

Here, $\underline{\underline{\mathbf{T}}}_n(\underline{q})$ are 6×6 dyadic functions, and $t_n(\underline{q})$ are scalar functions ($n=0,1,2,3,4$) of the unit spatial frequency vector $\underline{q}$. Furthermore, $\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q})$ may be partitioned as [13,22]

$$\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q})=\tilde{\underline{\underline{\mathbf{G}}}}^{0}_{BCM}(\underline{q})+\tilde{\underline{\underline{\mathbf{G}}}}^{\infty}_{BCM}(\hat{\underline{q}}), \tag{57}$$

where

$$\tilde{\underline{\underline{\mathbf{G}}}}^{\infty}_{BCM}(\hat{\underline{q}})=\lim_{q\to\infty}\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q})=\frac{1}{i\omega}\,\frac{\underline{\underline{\mathbf{T}}}_4(\hat{\underline{q}})}{t_4(\hat{\underline{q}})} \tag{58}$$

and (suppressing the dependences of $\underline{\underline{\mathbf{T}}}_n$ and t_n on $\underline{q}$)

$$\tilde{\underline{\underline{\mathbf{G}}}}^{0}_{BCM}(\underline{q})=\frac{(t_4\underline{\underline{\mathbf{T}}}_3-t_3\underline{\underline{\mathbf{T}}}_4)(q/\omega)^3+(t_4\underline{\underline{\mathbf{T}}}_2-t_2\underline{\underline{\mathbf{T}}}_4)(q/\omega)^2+(t_4\underline{\underline{\mathbf{T}}}_1-t_1\underline{\underline{\mathbf{T}}}_4)(q/\omega)+(t_4\underline{\underline{\mathbf{T}}}_0-t_0\underline{\underline{\mathbf{T}}}_4)}{i\omega t_4[t_4(q/\omega)^4+t_3(q/\omega)^3+t_2(q/\omega)^2+t_1(q/\omega)+t_0]}. \tag{59}$$

Explicit coordinate-free representations of $\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q})$ are already available for chiral [5], general dielectric [22], and dielectric-magnetic [23] mediums; we consider the structure of $\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{q})$ for reciprocal biaxial bianisotropic mediums in Sec. VI.

In order to advance our analysis, we have to specify the depolarization dyadic $\underline{\underline{\mathbf{D}}}$. That is best done by considering the topology of the composite, but unique answers are not possible in general [24]. Let $V^e_\eta(V^s_\eta)$ be an ellipsoidal (spherical) region, centered at the origin of our coordinate system, of size determined by the linear measure η. We imagine that both constituent phases are distributed as conformal ellipsoids of surfaces parametrized by

$$\underline{R}_e(\theta,\phi)=\eta\underline{\underline{U}}\cdot\hat{\underline{R}}(\theta,\phi), \tag{60}$$

where $\hat{\underline{R}}(\theta,\phi)$ is the radial unit vector depending on the spherical polar coordinates θ and ϕ, and $\underline{\underline{U}}$ is a real symmetric dyadic with positive eigenvalues a, b, and c. We mean here that both constituent phases are present with a distribution of η such that there is no vacant space in the composite medium. The same fractal-like topology is inherent in the Maxwell Garnett and Bruggeman formalisms, although it is rarely mentioned. Having selected $\underline{\underline{U}}$, we determine $\underline{\underline{\mathbf{D}}}$ as [13]

$$\underline{\underline{\mathbf{D}}}=\lim_{\delta\to 0}\int_{V^e_\delta}\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R})d^3\underline{R}=abc\lim_{\delta\to 0}\int_{V^s_\delta}\underline{\underline{\mathbf{G}}}_{BCM}(\underline{\underline{U}}\cdot\underline{H})d^3\underline{H}, \tag{61}$$

where the spherical region V^s_δ has radius δ, and $\underline{H}=\underline{\underline{U}}^{-1}\cdot\underline{R}$. We choose the covariance $\tau(\underline{R})$ to reflect the ellipsoidal topology relating to $\underline{\underline{D}}$. Accordingly,

$$\tau(\underline{R})=\begin{cases} f_af_b, & \underline{R}\in V^e_L\\ 0, & \underline{R}\notin V^e_L,\end{cases} \tag{62}$$

where L is the correlation length. Covariance functions in the form of step functions, of both isotropic [25,26] and anisotropic [3] types, have been considered in previous SPFT analyses.

Let $V^e_{L-\delta}=V^e_L-V^e_\delta$ and $V^s_{L-\delta}=V^s_L-V^s_\delta$. The principal-value integration in Eq. (52) now proceeds through the introduction of the Fourier transform of $\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R})$, facilitated by the changes of variable $\underline{H}=\underline{\underline{U}}^{-1}\cdot\underline{R}$ and $\underline{v}=\underline{\underline{U}}\cdot\underline{w}$ (where $\underline{H}$, $\underline{v}$, and $\underline{w}$ are dummy vector variables), as follows:

$$\begin{aligned}\frac{(2\pi)^3}{f_af_b}\mathrm{P}\int\ &\tau(\underline{R})\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R})d^3\underline{R}\\ &=\lim_{\delta\to 0}\int_{V^e_{L-\delta}}\left[\int_{\underline{w}}\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{w})\exp(i\underline{w}\cdot\underline{R})d^3\underline{w}\right]d^3\underline{R}\\ &=\lim_{\delta\to 0}\int_{\underline{v}}\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{\underline{U}}^{-1}\cdot\underline{v})\left[\int_{V^e_{L-\delta}}\exp(i\underline{v}\cdot\underline{H})d^3\underline{H}\right]d^3\underline{v}\\ &=\int_{\underline{v}}\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{\underline{U}}^{-1}\cdot\underline{v})\left[\frac{4\pi}{v^2}\left(\frac{\sin vL}{v}-L\cos vL\right)\right]d^3\underline{v}\\ &\quad-\lim_{\delta\to 0}\int_{\underline{v}}\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{\underline{U}}^{-1}\cdot\underline{v})\left[\frac{4\pi}{v^2}\left(\frac{\sin v\delta}{v}\right.\right.\\ &\quad\left.\left.-\delta\cos v\delta\right)\right]d^3\underline{v}.\end{aligned} \tag{63}$$

Now, from [22] we have

$$\underline{\underline{\mathbf{D}}}=\frac{1}{(2\pi)^3}\lim_{\delta\to 0}\int_{\underline{v}}\tilde{\underline{\underline{\mathbf{G}}}}_{BCM}(\underline{\underline{U}}^{-1}\cdot\underline{v})\times\left[\frac{4\pi}{v^2}\left(\frac{\sin v\delta}{v}-\delta\cos v\delta\right)\right]d^3\underline{v} \quad (64)$$

$$=\frac{1}{(2\pi)^3}\int_{\underline{v}}\tilde{\underline{\underline{\mathbf{G}}}}^{\infty}_{BCM}(\underline{\underline{U}}^{-1}\cdot\underline{v})\times\left[\frac{4\pi}{v^2}\left(\frac{\sin vE}{v}-E\cos vE\right)\right]d^3\underline{v} \quad (65)$$

for $E>0$. Thus, combining Eqs. (63)–(65) along with Eq. (57), we find

$$\mathrm{P}\int \tau(\underline{R})\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R})d^3\underline{R}=\frac{f_af_b}{2\pi^2}\int_{\phi_v=0}^{2\pi}\int_{\theta_v=0}^{\pi}\int_{v=0}^{\infty}\tilde{\underline{\underline{\mathbf{G}}}}^{0}_{BCM}(\underline{\underline{U}}^{-1}\cdot\underline{v})\times\left(\frac{\sin vL}{v}-L\cos vL\right)\sin\theta_v\,dv\,d\theta_v\,d\phi_v\,. \quad (66)$$

Although very cumbersome for reproduction here, a straightforward analysis shows that the determinant of $\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{w})$ is quadratic in w^2 for reciprocal bianisotropic mediums (i.e., $\underline{\underline{\epsilon}}=\underline{\underline{\epsilon}}^T$, $\underline{\underline{\xi}}=-\underline{\underline{\zeta}}^T$, and $\underline{\underline{\mu}}=\underline{\underline{\mu}}^T$) and general dielectric–magnetic mediums (i.e., $\underline{\underline{\xi}}=\underline{\underline{\zeta}}=\underline{\underline{0}}$). Consequently, provided the numerator of $\tilde{\underline{\underline{\mathbf{G}}}}^{0}_{BCM}(\underline{w})$ is an even function of w, the integration in Eq. (66) with respect to v may be evaluated by conventional calculus of residues for such mediums. However, for the general bianisotropic case, the determinant of $\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{w})$ is not an even function of w and alternative methods may be required to compute the integral (66).

In addition to being central to the calculation of $\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{0})$, the integral (66) provides a correction to the depolarization dyadic $\underline{\underline{\mathbf{D}}}$ when the exclusion region V_L^e is of finite size. This is also reflected in the fact that $\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{0})$ represents a modification of $\underline{\underline{\mathbf{K}}}_{BCM}$; indeed,

$$\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{0})=\underline{\underline{\mathbf{K}}}_{BCM}-\frac{1}{i\omega}[\underline{\underline{\mathbf{I}}}+\tilde{\underline{\underline{\mathbf{\Sigma}}}}(\underline{0})\cdot\underline{\underline{\mathbf{D}}}]^{-1}\cdot\tilde{\underline{\underline{\mathbf{\Sigma}}}}(\underline{0}). \quad (67)$$

This key equation is the focus of the remainder of this paper.

VI. IMPLEMENTATION OF THE LONG-WAVELENGTH APPROXIMATION

In order to illustrate the implementation of the long-wavelength approximation in the bilocal SPFT framework, we consider a two-phase composite for which both constituent phases belong to the general class of reciprocal biaxial bianisotropic mediums. The constitutive dyadics $\underline{\underline{\epsilon}}_p$, $\underline{\underline{\xi}}_p=-\underline{\underline{\zeta}}_p$, and $\underline{\underline{\mu}}_p$ of the constituent phases are taken to have the same eigenvectors, i.e.,

$$\underline{\underline{\epsilon}}_p=\begin{bmatrix}\epsilon_x^p & 0 & 0\\ 0 & \epsilon_y^p & 0\\ 0 & 0 & \epsilon_z^p\end{bmatrix},\quad \underline{\underline{\xi}}_p=\begin{bmatrix}\xi_x^p & 0 & 0\\ 0 & \xi_y^p & 0\\ 0 & 0 & \xi_z^p\end{bmatrix}=-\underline{\underline{\zeta}}_p,$$

$$\underline{\underline{\mu}}_p=\begin{bmatrix}\mu_x^p & 0 & 0\\ 0 & \mu_y^p & 0\\ 0 & 0 & \mu_z^p\end{bmatrix}, \quad (68)$$

where all diagonal entries are complex valued. Thorough reinvestigations of the correct formulation of constitutive relations for biaxial mediums were recently provided [27,28]. For simplicity, we choose a spherical particulate topology for the constituent mediums, i.e., $\underline{\underline{U}}=\underline{\underline{I}}$. Analytical as well as numerical results for ellipsoidal particulate topology will be presented in detail in future publications.

As emphasized in Sec. V, the crucial step in applying the long-wavelength approximation is the calculation of the volume integral (66). We now proceed to evaluate the integration with respect to v in Eq. (66) by means of residue calculus, exploiting symmetries in the integrand along the way. We begin by considering the singularities of $\tilde{\underline{\underline{\mathbf{G}}}}^{0}_{BCM}(\underline{v})$, i.e., the zeros of $t_4(\hat{\underline{v}})\det\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{v})$. Taking the determinant of Eq. (55), we find

$$\det\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{v})=t_4(\hat{\underline{v}})(v/\omega)^4+t_2(\hat{\underline{v}})(v/\omega)^2+t_0, \quad (69)$$

$$t_4(\hat{\underline{v}})=(\hat{\underline{v}}\cdot\underline{\underline{\epsilon}}_{BCM}\cdot\hat{\underline{v}})(\hat{\underline{v}}\cdot\underline{\underline{\mu}}_{BCM}\cdot\hat{\underline{v}})+(\hat{\underline{v}}\cdot\underline{\underline{\xi}}_{BCM}\cdot\hat{\underline{v}})\times(\hat{\underline{v}}\cdot\underline{\underline{\xi}}_{BCM}\cdot\hat{\underline{v}}), \quad (70)$$

$$\begin{aligned}t_2(\hat{\underline{v}})=\mathrm{tr}\{&[\hat{\underline{v}}\times\mathrm{adj}(\underline{\underline{\epsilon}}_{BCM})]\cdot[\hat{\underline{v}}\times\mathrm{adj}(\underline{\underline{\mu}}_{BCM})]\\&-[\hat{\underline{v}}\times\mathrm{adj}(\underline{\underline{\xi}}_{BCM})]\cdot[\hat{\underline{v}}\times\mathrm{adj}(\underline{\underline{\xi}}_{BCM})]\}\\&+\hat{\underline{v}}\cdot[\underline{\underline{\epsilon}}_{BCM}\cdot\underline{\underline{F}}(\underline{\underline{\xi}}_{BCM},\underline{\underline{\xi}}_{BCM})\cdot\underline{\underline{\mu}}_{BCM}\\&-\underline{\underline{\xi}}_{BCM}\cdot\underline{\underline{F}}(\underline{\underline{\epsilon}}_{BCM},\underline{\underline{\mu}}_{BCM})\cdot\underline{\underline{\xi}}_{BCM}]\cdot\hat{\underline{v}},\end{aligned} \quad (71)$$

$$t_0=\det(\underline{\underline{\Omega}}), \quad (72)$$

$$\underline{\underline{F}}(\underline{\underline{m}},\underline{\underline{n}})=[(\mathrm{tr}\,\underline{\underline{m}})\underline{\underline{\mathbf{I}}}-\underline{\underline{m}}]\cdot[(\mathrm{tr}\,\underline{\underline{n}})\underline{\underline{\mathbf{I}}}-\underline{\underline{n}}]-[(\mathrm{tr}\,\underline{\underline{m}}\cdot\underline{\underline{n}})\underline{\underline{\mathbf{I}}}-\underline{\underline{m}}\cdot\underline{\underline{n}}], \quad (73)$$

$$\underline{\underline{\Omega}}=\underline{\underline{\epsilon}}_{BCM}\cdot\underline{\underline{\mu}}_{BCM}+\underline{\underline{\xi}}_{BCM}\cdot\underline{\underline{\xi}}_{BCM}, \quad (74)$$

where $\underline{\underline{m}}$ and $\underline{\underline{n}}$ are arbitrary 3×3 dyadics. The issues of the location and nature of the singularities of $\tilde{\underline{\underline{\mathbf{G}}}}^{0}_{BCM}(\underline{v})$ are rather involved in the most general setting; see Cottis and Kondylis [29] for a detailed discussion in the case of an anisotropic dielectric medium. We refrain from considering pathological special cases here: that is, for all values of $\hat{\underline{v}}$ we assume that (i) $t_4(\hat{\underline{v}})\neq 0$, and (ii) the v^2 roots of $\det\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{v})$ are distinct, i.e., $\kappa_+\neq\kappa_-$, where

$$\kappa_{\pm}=\omega^2\left(\frac{-t_2(\hat{\underline{v}})\pm\sqrt{t_2^2(\hat{\underline{v}})-4t_4(\hat{\underline{v}})t_0}}{2t_4(\hat{\underline{v}})}\right). \quad (75)$$

We note that the possibility $\kappa_+=\kappa_-$ is most likely to arise when we have isotropic constituent phases combined with spherical topology, but SPFT in this instance has been treated by Michel and Lakhtakia [5].

With the foregoing simplifications in place, we find that the singularities of $\tilde{\underline{\underline{\mathbf{G}}}}^0_{BCM}(\underline{v})$ occur as simple poles at

$$v=\sqrt{\kappa_+},-\sqrt{\kappa_+},\sqrt{\kappa_-},-\sqrt{\kappa_-}. \quad (76)$$

Therefore, for definiteness, we take both $\sqrt{\kappa_+}$ and $\sqrt{\kappa_-}$ to lie in the upper half of the complex v plane (inclusive of the real axis).

We next turn our attention to symmetries of $\tilde{\underline{\underline{\mathbf{G}}}}^0_{BCM}(\underline{v})$. For reciprocal biaxial bianisotropic mediums, the numerator of $\tilde{\underline{\underline{\mathbf{G}}}}^0_{BCM}(\underline{v})$ is a dyadic function that is cubic in v. However, by a straightforward—albeit lengthy—utilization of Eq. (55), we find that the odd terms in the numerator of $\tilde{\underline{\underline{\mathbf{G}}}}^0_{BCM}(\underline{v})$ integrate to zero with respect to the angular variables in Eq. (66). The remaining even terms $\underline{\underline{\mathbf{T}}}_n(\hat{\underline{v}})$ $(n=0,2,4)$ in the notation of Eq. (59) are conveniently expressed in terms of four 3×3 dyadics $\underline{\underline{T}}_n^{\lambda}(\hat{\underline{v}})$ $(\lambda=ee,em,me,mm)$ as follows:

$$\underline{\underline{\mathbf{T}}}_n(\hat{\underline{v}})=\begin{bmatrix}\underline{\underline{T}}_n^{ee}(\hat{\underline{v}}) & \underline{\underline{T}}_n^{em}(\hat{\underline{v}})\\ \underline{\underline{T}}_n^{me}(\hat{\underline{v}}) & \underline{\underline{T}}_n^{mm}(\hat{\underline{v}})\end{bmatrix}, \quad n=0,2,4. \quad (77)$$

By symmetry considerations, only the diagonal entries of $\underline{\underline{T}}_n^{\lambda}$ give rise to nonzero integrals in Eq. (66). Thus, in evaluating Eq. (66), we can replace $\tilde{\underline{\underline{\mathbf{G}}}}^0_{BCM}(\underline{v})$ by

$$\frac{1}{i\omega}\left(\frac{\underline{\underline{\boldsymbol{\alpha}}}(\hat{\underline{v}})(v/\omega)^2+\underline{\underline{\beta}}(\hat{\underline{v}})}{t_4(\hat{\underline{v}})(v/\omega)^4+t_2(\hat{\underline{v}})(v/\omega)^2+t_0}\right), \quad (78)$$

where we have

$$[\underline{\underline{\boldsymbol{\alpha}}}(\hat{\underline{v}})]_{lj}=\begin{cases}[\underline{\underline{\mathbf{T}}}_2(\hat{\underline{v}})]_{lj}-\dfrac{t_2(\hat{\underline{v}})}{t_4(\hat{\underline{v}})}[\underline{\underline{\mathbf{T}}}_4(\hat{\underline{v}})]_{lj}, & l(\mathrm{mod}3)\equiv j(\mathrm{mod}3)\\ 0, & l(\mathrm{mod}3)\not\equiv j(\mathrm{mod}3)\end{cases} \quad (79)$$

and

$$[\underline{\underline{\beta}}(\hat{\underline{v}})]_{lj}=\begin{cases}[\underline{\underline{\mathbf{T}}}_0(\hat{\underline{v}})]_{lj}-\dfrac{t_0}{t_4(\hat{\underline{v}})}[\underline{\underline{\mathbf{T}}}_4(\hat{\underline{v}})]_{lj}, & l(\mathrm{mod}3)\equiv j(\mathrm{mod}3)\\ 0, & l(\mathrm{mod}3)\not\equiv j(\mathrm{mod}3)\end{cases} \quad (80)$$

for $l,j=1,2,\ldots,6$, and $[\underline{\underline{\mathbf{T}}}_n]_{lj}$ denotes the ljth entry of the 6×6 dyadic $\underline{\underline{\mathbf{T}}}_n$.

The dyadics $\underline{\underline{\mathbf{T}}}_0(\hat{\underline{v}})$ and $\underline{\underline{\mathbf{T}}}_4(\hat{\underline{v}})$ are readily extracted from the adjoint of $\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{v})$. They are given in coordinate-free form as

$$\underline{\underline{\mathbf{T}}}_0(\hat{\underline{v}})=\begin{bmatrix}\underline{\underline{\mu}}_{BCM}\cdot\mathrm{adj}(\underline{\underline{\Omega}}) & -\underline{\underline{\xi}}_{BCM}\cdot\mathrm{adj}(\underline{\underline{\Omega}})\\ \underline{\underline{\xi}}_{BCM}\cdot\mathrm{adj}(\underline{\underline{\Omega}}) & \underline{\underline{\epsilon}}_{BCM}\cdot\mathrm{adj}(\underline{\underline{\Omega}})\end{bmatrix} \quad (81)$$

and

$$\underline{\underline{\mathbf{T}}}_4(\hat{\underline{v}})=\begin{bmatrix}(\hat{\underline{v}}\cdot\underline{\underline{\mu}}_{BCM}\cdot\hat{\underline{v}})\hat{\underline{v}}\hat{\underline{v}} & -(\hat{\underline{v}}\cdot\underline{\underline{\xi}}_{BCM}\cdot\hat{\underline{v}})\hat{\underline{v}}\hat{\underline{v}}\\ (\hat{\underline{v}}\cdot\underline{\underline{\xi}}_{BCM}\cdot\hat{\underline{v}})\hat{\underline{v}}\hat{\underline{v}} & (\hat{\underline{v}}\cdot\underline{\underline{\epsilon}}_{BCM}\cdot\hat{\underline{v}})\hat{\underline{v}}\hat{\underline{v}}\end{bmatrix}. \quad (82)$$

The dyadic $\underline{\underline{\mathbf{T}}}_2(\hat{\underline{v}})$ has a more complex structure, but only its diagonal components are needed to evaluate Eq. (66),

$$[\underline{\underline{T}}_2^{ee}(\hat{\underline{v}})]_{ll}=[\{2\underline{\underline{\epsilon}}\cdot\mathrm{adj}(\underline{\underline{\mu}})-\mathrm{tr}[\underline{\underline{\epsilon}}\cdot\mathrm{adj}(\underline{\underline{\mu}})]\underline{\underline{I}}+\underline{\underline{\mu}}\cdot\underline{\underline{F}}(\underline{\underline{\xi}},\underline{\underline{\xi}})\}\cdot\hat{\underline{v}}\hat{\underline{v}} -\underline{\underline{F}}(\underline{\underline{\mu}},\underline{\underline{\xi}}\cdot\underline{\underline{\xi}}\cdot\hat{\underline{v}}\hat{\underline{v}})-(\hat{\underline{v}}\cdot\underline{\underline{\epsilon}}\cdot\hat{\underline{v}})\mathrm{adj}(\underline{\underline{\mu}})]_{ll}, \quad l=1,2,3 \quad (83)$$

$$[\underline{\underline{T}}_2^{em}(\hat{\underline{v}})]_{ll}=[[\underline{\underline{F}}(\underline{\underline{\epsilon}},\underline{\underline{\mu}})\cdot\underline{\underline{\xi}}-\det(\underline{\underline{\xi}})\underline{\underline{I}}]\cdot\hat{\underline{v}}\hat{\underline{v}}-\underline{\underline{F}}(\underline{\underline{\epsilon}},\underline{\underline{\mu}}\cdot\hat{\underline{v}}\hat{\underline{v}},\underline{\underline{\xi}}) -(\hat{\underline{v}}\cdot\underline{\underline{\xi}}\cdot\hat{\underline{v}})\mathrm{adj}(\underline{\underline{\xi}})]_{ll}, \quad l=1,2,3 \quad (84)$$

$$[\underline{\underline{T}}_2^{me}(\hat{\underline{v}})]_{ll}=-[[\underline{\underline{F}}(\underline{\underline{\epsilon}},\underline{\underline{\mu}})\cdot\underline{\underline{\xi}}-\det(\underline{\underline{\xi}})\underline{\underline{I}}]\cdot\hat{\underline{v}}\hat{\underline{v}}-\underline{\underline{F}}(\underline{\underline{\epsilon}}\cdot\underline{\underline{\mu}}\cdot\hat{\underline{v}}\hat{\underline{v}},\underline{\underline{\xi}}) -(\hat{\underline{v}}\cdot\underline{\underline{\xi}}\cdot\hat{\underline{v}})\mathrm{adj}(\underline{\underline{\xi}})]_{ll}, \quad l=1,2,3 \quad (85)$$

$$[\underline{\underline{T}}_2^{mm}(\hat{\underline{v}})]_{ll}=[\{2\,\mathrm{adj}(\underline{\underline{\epsilon}})\cdot\underline{\underline{\mu}}-\mathrm{tr}[\mathrm{adj}(\underline{\underline{\epsilon}})\cdot\underline{\underline{\mu}}]\underline{\underline{I}}+\underline{\underline{\epsilon}}\cdot\underline{\underline{F}}(\underline{\underline{\xi}},\underline{\underline{\xi}})\}\cdot\hat{\underline{v}}\hat{\underline{v}} -\underline{\underline{F}}(\underline{\underline{\epsilon}},\underline{\underline{\xi}}\cdot\underline{\underline{\xi}}\cdot\hat{\underline{v}}\hat{\underline{v}})-(\hat{\underline{v}}\cdot\underline{\underline{\mu}}\cdot\hat{\underline{v}})\mathrm{adj}(\underline{\underline{\epsilon}})]_{ll}, \quad l=1,2,3. \quad (86)$$

Residue calculus thereby results in the following evaluation of the integral with respect to v in Eq. (66):

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{v=0}^{\infty}\tilde{\underline{\underline{\mathbf{G}}}}^{0}_{BCM}(\underline{v})\left(\frac{\sin vL}{v}-L\cos vL\right)\sin\theta\, dv\, d\theta\, d\phi$$
$$=\frac{\pi\omega^3}{2i}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{t_4(\hat{\underline{v}})}\left\{\frac{1}{\kappa_+-\kappa_-}\left[e^{iLv}(1-iLv)\left(\frac{\underline{\underline{\alpha}}(\hat{\underline{v}})}{\omega^2}+\frac{\underline{\underline{\beta}}(\hat{\underline{v}})}{v^2}\right)\right]_{v=\sqrt{\kappa_-}}^{v=\sqrt{\kappa_+}}+\frac{\underline{\underline{\beta}}(\hat{\underline{v}})}{\kappa_+\kappa_-}\right\}\sin\theta\, d\theta\, d\phi. \tag{87}$$

Thus, Eq. (66) reduces to a surface integral which, in general, must be handled numerically.

An expression equivalent to Eq. (87) can be developed without explicit reference to the components $\underline{\underline{\mathbf{T}}}_n(\hat{\underline{v}})$ of the adjoint dyadic of $\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{v})$. This derivation—though less instructive—leads to a surface integral more amenable to numerical evaluation than that of Eq. (87). Introducing

$$\underline{\underline{\mathbf{N}}}(\underline{v})=\frac{\mathrm{adj}[\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{v})]-\det[\tilde{\underline{\underline{\mathbf{A}}}}_{BCM}(\underline{v})]\tilde{\underline{\underline{\mathbf{G}}}}^{\infty}_{BCM}(\hat{\underline{v}})}{(\hat{\underline{v}}\cdot\underline{\underline{\epsilon}}_{BCM}\cdot\hat{\underline{v}})(\hat{\underline{v}}\cdot\underline{\underline{\mu}}_{BCM}\cdot\hat{\underline{v}})+(\hat{\underline{v}}\cdot\underline{\underline{\xi}}_{BCM}\cdot\hat{\underline{v}})(\hat{\underline{v}}\cdot\underline{\underline{\xi}}_{BCM}\cdot\hat{\underline{v}})}, \tag{88}$$

we find that

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{v=0}^{\infty}\tilde{\underline{\underline{\mathbf{G}}}}^{0}_{BCM}(\underline{v})\left(\frac{\sin vL}{v}-L\cos vL\right)\sin\theta\, dv\, d\theta\, d\phi$$
$$=\frac{\pi\omega^3}{4i}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left\{\frac{1}{\kappa_+-\kappa_-}\left[\frac{e^{iLv}}{v^2}(1-iLv)[\underline{\underline{\mathbf{N}}}(\underline{v})+\underline{\underline{\mathbf{N}}}(-\underline{v})]\right]_{v=\sqrt{\kappa_-}}^{v=\sqrt{\kappa_+}}+\frac{2\underline{\underline{\mathbf{N}}}(\underline{0})}{\kappa_+\kappa_-}\right\}\sin\theta\, d\theta\, d\phi. \tag{89}$$

The specification of the bilocal SPFT equations in the long-wavelength approximation for reciprocal biaxial bianisotropic composites with spherical topology is completed by the following expression for the corresponding depolarization dyadic [13]:

$$\underline{\underline{\mathbf{D}}}=\frac{1}{4\pi i\omega}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{t_4(\hat{\underline{v}})}\begin{bmatrix}(\hat{\underline{v}}\cdot\underline{\underline{\mu}}_{BCM}\cdot\hat{\underline{v}})\hat{\underline{v}}\hat{\underline{v}} & -(\hat{\underline{v}}\cdot\underline{\underline{\xi}}_{BCM}\cdot\hat{\underline{v}})\hat{\underline{v}}\hat{\underline{v}}\\ (\hat{\underline{v}}\cdot\underline{\underline{\xi}}_{BCM}\cdot\hat{\underline{v}})\hat{\underline{v}}\hat{\underline{v}} & (\hat{\underline{v}}\cdot\underline{\underline{\epsilon}}_{BCM}\cdot\hat{\underline{v}})\hat{\underline{v}}\hat{\underline{v}}\end{bmatrix}\sin\theta\, d\theta\, d\phi. \tag{90}$$

Thus, the constitutive dyadic $\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{0})$ of the local effective medium is fully specified through Eqs. (67), (52), (87), (89), and (90).

Finally, in this section we consider the long-wavelength regime $|L\sqrt{\kappa_\pm}|\ll 1$. Retaining terms only of $O(L\sqrt{\kappa_\pm})$, we see that the integrals (87) and (89) become null valued. This is consistent with the finding reported for chiral mediums that the bilocal-approximated SPFT does not yield a correction to the Bruggeman formalism in the static limit [25]. However, on retaining terms up to $O(L^2\kappa_\pm)$, the $e^{iLv}(1-iLv)$ term in the integrand of Eqs. (87) and (89) reduces to $1+(Lv)^2/2$; hence the principal-value integration (66) becomes

$$\mathrm{P}\int\tau(\underline{R})\underline{\underline{\mathbf{G}}}_{BCM}(\underline{R})d^3\underline{R}$$
$$=\frac{-i\omega f_a f_b L^2}{8\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{\underline{\underline{\alpha}}(\hat{\underline{v}})}{t_4(\hat{\underline{v}})}\sin\theta\, d\theta\, d\phi. \tag{91}$$

For constituent mediums with real-valued constitutive parameters $\underline{\underline{\mathbf{K}}}_p$, the integrand of Eq. (91) is correspondingly real valued. Thus, it is clear from Eqs. (67), (52), (25), and (90) that scattering losses cannot be predicted under the long-wavelength approximation represented by Eq. (91). A similar result was reported for SPFT applied to chiral mediums [25].

VII. ILLUSTRATIVE NUMERICAL RESULTS

Since a detailed numerical study is the focus of a future paper [14], here we present numerical results for only two classes of bianisotropic composites—for illustration only. The first class allows direct comparison of results with earlier SPFT analyses, namely, the case involving chiral constituent phases. The second class comprises the more general reciprocal biaxial bianisotropic composites. An angular frequency ω of $2\pi\times10^{10}$ rad s^{-1} was used for all calculations reported here.

A. Chiral constituent phases

To allow direct comparison with the SPFT analyses of Michel and Lakhtakia [5,26], we choose the phase a to be a chiral medium described by the constitutive dyadic

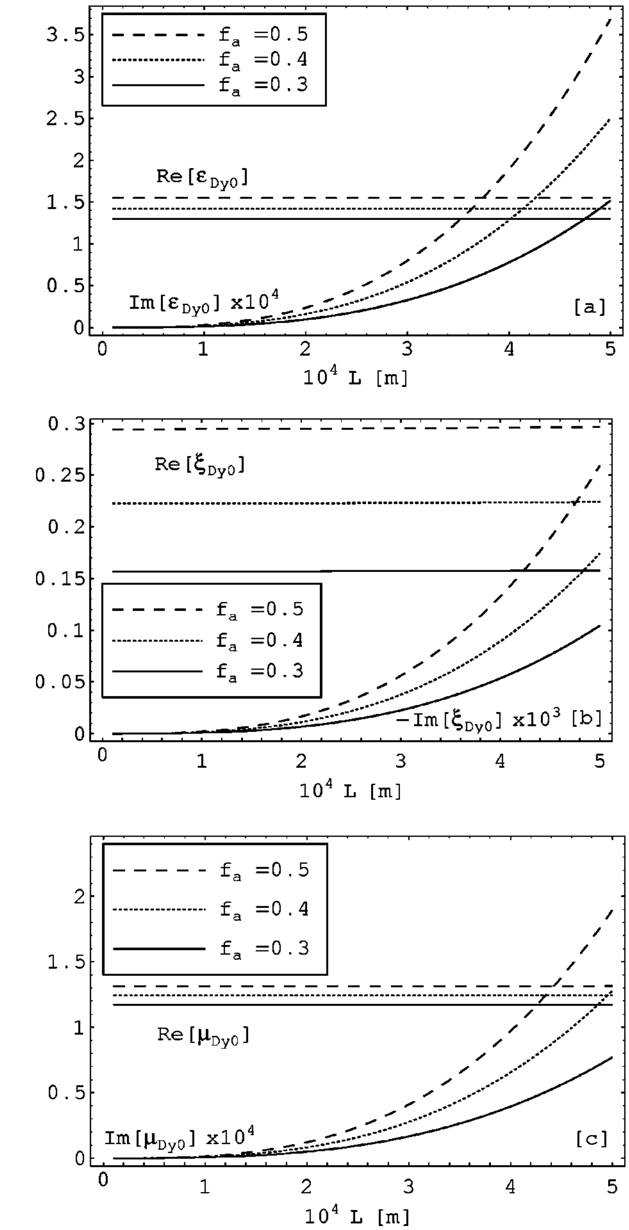

FIG. 1. Effective constitutive scalars of an isotropic chiral composite plotted as functions of the correlation length L for $f_a = 0.3$, 0.4, and 0.5. See Sec. VII A for $\underline{\underline{\mathbf{K}}}_a$, $\underline{\underline{\mathbf{K}}}_b$, and $\underline{\underline{\tilde{\mathbf{K}}}}_{Dy}(\underline{0})$.

$$\underline{\underline{\mathbf{K}}}_a = \begin{bmatrix} \epsilon_0 \epsilon_a \underline{\underline{I}} & i\sqrt{\epsilon_0 \mu_0}\,\xi_a \underline{\underline{I}} \\ -i\sqrt{\epsilon_0 \mu_0}\,\xi_a \underline{\underline{I}} & \mu_0 \mu_a \underline{\underline{I}} \end{bmatrix}, \quad (92)$$

where $\epsilon_a = 2.304$, $\xi_a = 0.724$, and $\mu_a = 1.728$, with $\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$ and $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}$ being the permittivity and permeability of free space (i.e., vacuum), respectively. The phase b is simply taken to be free space itself. For this isotropic example, the effective medium is characterized by the constitutive dyadic

$$\underline{\underline{\tilde{\mathbf{K}}}}_{Dy}(\underline{0}) = \begin{bmatrix} \epsilon_0 \epsilon_{Dy0} \underline{\underline{I}} & i\sqrt{\epsilon_0 \mu_0}\,\xi_{Dy0} \underline{\underline{I}} \\ -i\sqrt{\epsilon_0 \mu_0}\,\xi_{Dy0} \underline{\underline{I}} & \mu_0 \mu_{Dy0} \underline{\underline{I}} \end{bmatrix}. \quad (93)$$

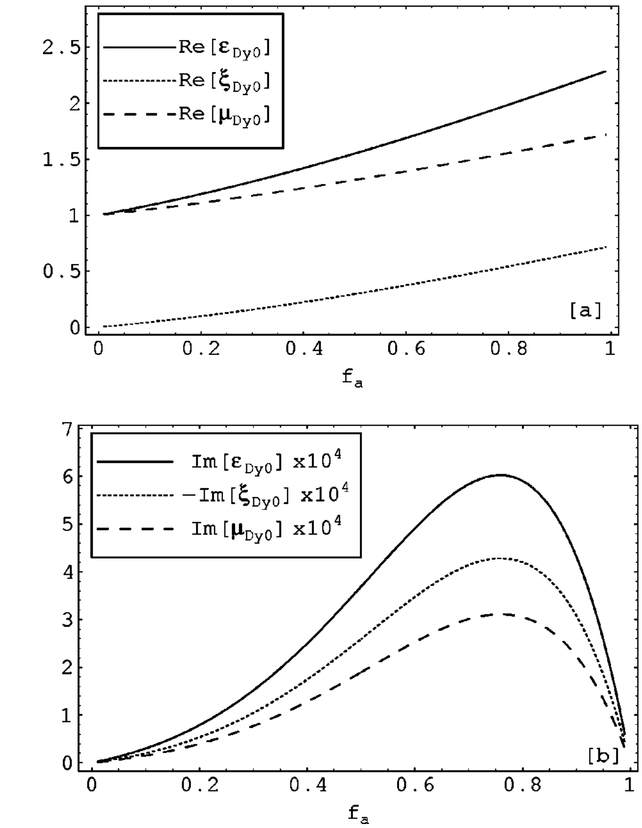

FIG. 2. (a) Real and (b) imaginary parts of the effective constitutive scalars of an isotropic chiral composite plotted as functions f_a for a correlation length $L = 5 \times 10^{-4}$ m. See Sec. VII A for $\underline{\underline{\mathbf{K}}}_a$, $\underline{\underline{\mathbf{K}}}_b$, and $\underline{\underline{\tilde{\mathbf{K}}}}_{Dy}(\underline{0})$.

The computed values of ϵ_{Dy0}, ξ_{Dy0}, and μ_{Dy0} are plotted as functions of correlation length L in Fig. 1 for three different values of volume fraction f_a. All three effective constitutive scalars exhibit a similar dependency on L: their imaginary parts increase sharply from zero as the correlation length increases from zero, whereas their real parts remain almost constant. Upon converting from the present Tellegen notation to the Drude-Born-Federov notation [21], the computed values of the effective constitutive scalars at $f_a = 0.3$ are found to be in complete agreement with those values calculated previously [26]. We note that the derivation presented by Michel and Lakhtakia [5] proceeded in a somewhat different manner to the one here, as an explicit expression for the dyadic Green function is available for chiral mediums. Over the range $0.3 \leqslant f_a \leqslant 0.5$, the graphs in Fig. 1 clearly demonstrate that the degree of attenuation (due to scattering losses) increases as the volumetric proportion of phase a increases. Attenuation is also predicted in the *extended* Maxwell Garnett and Bruggeman formalisms [30–32] when the finite size of inclusions is explicitly considered, but the correlation length is not relevant to those formalisms.

This issue is pursued further in Fig. 2 where, for a fixed correlation length $L = 5 \times 10^{-4}$ m, the constitutive scalars of the effective medium are plotted as functions of f_a. The maximum degree of attenuation, as indicated by the magnitude of the imaginary parts of the effective constitutive sca-

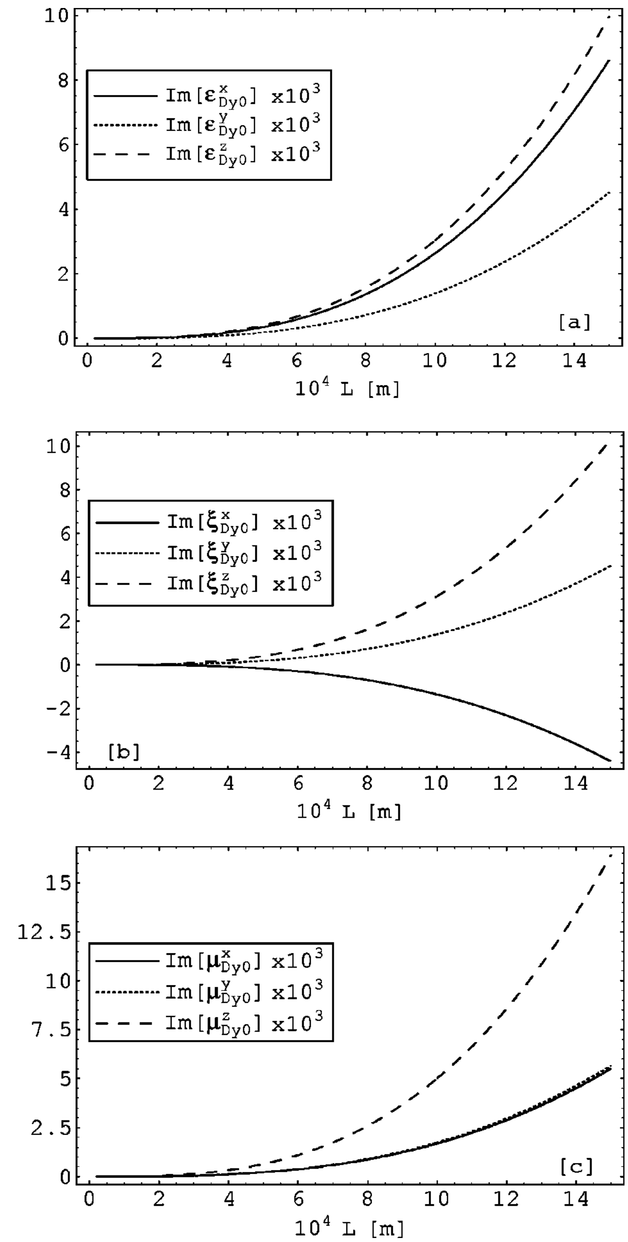

FIG. 3. Imaginary parts of the effective constitutive scalars of a reciprocal bianisotropic composite plotted as functions of the correlation length L for $f_a = 0.3$. See Sec. VII B for $\underline{\underline{\mathbf{K}}}_a$, $\underline{\underline{\mathbf{K}}}_b$, and $\underline{\underline{\tilde{\mathbf{K}}}}_{Dy}(\underline{0})$.

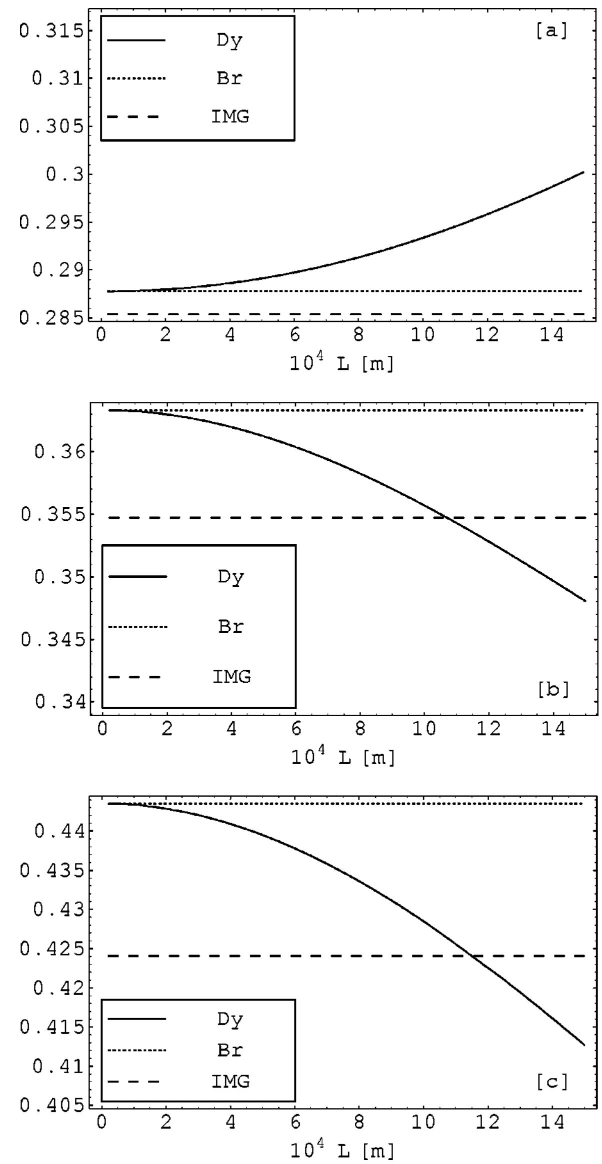

FIG. 4. Real parts of (a) ξ^x_{Dy0}, (b) ξ^y_{Dy0}, and (c) ξ^z_{Dy0} of a reciprocal bianisotropic composite plotted as functions of the correlation length L for $f_a = 0.3$. The L-independent values computed using the Bruggeman and incremental Maxwell Garnett formalisms are also presented. See Sec. VII B for $\underline{\underline{\mathbf{K}}}_a$, $\underline{\underline{\mathbf{K}}}_b$, and $\underline{\underline{\tilde{\mathbf{K}}}}_{Dy}(\underline{0})$.

lars, occurs at $f_a \approx 0.75$. Beyond this value of f_a, fewer scattering centers are present in a composite. The real parts of the effective constitutive scalars follow an almost linear progression between the values they must hold take at $f_a = 0$ and $f_a = 1$.

B. Biaxial constituent phases

For phase a we again choose a chiral medium: in the notation of Eq. (92), we take $\epsilon_a = 2$, $\xi_a = 1$, and $\mu_a = 1.5$. For phase b we select the biaxial dielectric-magnetic medium specified by

$$\underline{\underline{K}}_b = \begin{bmatrix} \epsilon_0 \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 3.5 \end{pmatrix} & \underline{\underline{0}} \\ \underline{\underline{0}} & \mu_0 \begin{pmatrix} 1.4 & 0 & 0 \\ 0 & 2.4 & 0 \\ 0 & 0 & 3.4 \end{pmatrix} \end{bmatrix}. \tag{94}$$

The effective medium arising from these constituent phases has a reciprocal biaxial bianisotropic structure characterized by the following constitutive dyadic:

$$\tilde{\underline{\underline{K}}}_{Dy}(\underline{0}) = \begin{bmatrix} \epsilon_0 \begin{pmatrix} \epsilon^x_{Dy0} & 0 & 0 \\ 0 & \epsilon^y_{Dy0} & 0 \\ 0 & 0 & \epsilon^z_{Dy0} \end{pmatrix} & i\sqrt{\epsilon_0\mu_0} \begin{pmatrix} \xi^x_{Dy0} & 0 & 0 \\ 0 & \xi^y_{Dy0} & 0 \\ 0 & 0 & \xi^z_{Dy0} \end{pmatrix} \\ -i\sqrt{\epsilon_0\mu_0} \begin{pmatrix} \xi^x_{Dy0} & 0 & 0 \\ 0 & \xi^y_{Dy0} & 0 \\ 0 & 0 & \xi^z_{Dy0} \end{pmatrix} & \mu_0 \begin{pmatrix} \mu^x_{Dy0} & 0 & 0 \\ 0 & \mu^y_{Dy0} & 0 \\ 0 & 0 & \mu^z_{Dy0} \end{pmatrix} \end{bmatrix}. \tag{95}$$

The imaginary parts of the constitutive scalars appearing on the right side of Eq. (95) are plotted as functions of correlation length L in Fig. 3. As in the isotropic case illustrated in Fig. 1, the magnitude of the imaginary parts increases sharply with increasing L. The real parts of the constitutive scalars are largely insensitive to the correlation length.

Computed values of the real components of the effective magnetoelectric scalars ξ^x_{Dy0}, ξ^y_{Dy0}, and ξ^z_{Dy0} are displayed in Fig. 4 as functions of L. For comparison, the corresponding (constant) values computed by the Bruggeman and incremental Maxwell Garnett (IMG) formalisms are also presented. The IMG results were generated using five incremental steps [15,16]. The SPFT-estimated values are seen to coincide with the Bruggeman estimates at $L=0$, but increasingly deviate from them as L increases. Note that both the Bruggeman and IMG formalisms do not predict scattering losses in the effective medium when the constituent phases are nondissipative, as is the case here. Analogous comparisons for the effective dielectric and magnetic scalars in $\tilde{\underline{\underline{\mathbf{K}}}}_{Dy}(\underline{0})$ are not presented here as they behave similarly to the magnetoelectric scalars graphed in Fig. 4. To conclude, further numerical results will be presented in detail in a future paper [14].

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[1] *Selected Papers on Linear Optical Composite Materials*, edited by A. Lakhtakia (SPIE Optical Engineering Press, Bellingham, WA, 1996).
[2] A. Lakhtakia, Microwave Opt. Technol. Lett. **25**, 53 (2000).
[3] L. Tsang and J. A. Kong, Radio Sci. **16**, 303 (1981).
[4] N. P. Zhuck, Phys. Rev. B **50**, 15 636 (1994).
[5] B. Michel and A. Lakhtakia, Phys. Rev. E **51**, 5701 (1995).
[6] U. B. Unrau, in *Proceedings of Chiral 94: 3rd International Workshop on Chiral, Bi-isotropic and Bi-anisotropic Media*, edited by F. Mariotte and J.-P. Parneix (French Atomic Energy Commission, Le Barp, France, 1994), Appendix.
[7] *Advances in Complex Electromagnetic Materials*, edited by A. Priou, A. Shivola, S. Tretyakov, and A. Vinogradov (Kluwer, Dordrecht, 1997).
[8] *Proceedings of Bianisotropics '97: International Conference and Workshop on Electromagnetics of Complex Media*, edited by W. S. Weiglhofer (University of Glasgow, Glasgow, UK, 1997).
[9] *Proceedings of Bianisotropics '98: 7th International Conferences on Complex Media*, edited by A. F. Jacob and J. Reichert (Technische Universität Braunschweig, Braunschweig, Germany, 1998).
[10] B. Michel, A. Lakhtakia, and W. S. Weiglhofer, Int. J. Appl. Electromagn. Mech. **9**, 167 (1998); **10**, 537(E) (1999).
[11] V. V. Tamoikin, Radiophys. Quantum Electron. **14**, 228 (1971).
[12] U. Frisch, in *Probabilistic Methods in Applied Mathematics*, edited by A. T. Bharucha-Reid (Academic Press, London, 1970), Vol. 1.
[13] B. Michel and W. S. Weiglhofer, Arch. Elektr. Uebertrag. **51**, 219 (1997); **52**, 31(E) (1998).
[14] Tom G. Mackay, Akhlesh Lakhtakia, and Werner S. Weiglhofer, Phys. Rev. E (to be published).
[15] A. Lakhtakia, Microwave Opt. Technol. Lett. **17**, 276 (1998).
[16] B. Michel, A. Lakhtakia, W. S. Weiglhofer, and T. G. Mackay, Compos. Sci. Technol. (to be published).
[17] H. Hochstadt, *Integral Equations* (Wiley, New York, 1973).
[18] W. S. Weiglhofer, A. Lakhtakia, and B. Michel, Microwave Opt. Technol. Lett. **15**, 263 (1997); **22**, 221(E) (1999).
[19] A. Stogryn, IEEE Trans. Antennas Propag. **31**, 985 (1983).
[20] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, 4th ed. (Academic Press, London, 1995).
[21] A. Lakhtakia, *Beltrami Fields in Chiral Media* (World Scientific, Singapore, 1994).
[22] B. Michel, Int. J. Appl. Electromagn. Mech. **8**, 219 (1997).
[23] N. P. Zhuck and A. S. Omar, IEEE Trans. Antennas Propag. **47**, 1364 (1999).
[24] A. Lakhtakia, B. Michel, and W. S. Weiglhofer, Compos. Sci. Technol. **57**, 185 (1997).
[25] B. Michel and A. Lakhtakia, J. Phys. D: Appl. Phys. **29**, 1431 (1996).
[26] B. Michel and A. Lakhtakia, J. Phys. D: Appl. Phys. **32**, 404 (1999).
[27] W. S. Weiglhofer and A. Lakhtakia, Electromagnetics **19**, 351 (1999).

[28] T. G. Mackay and W. S. Weiglhofer, J. Opt. A: Pure Appl. Opt. **2**, 426 (2000).
[29] P. G. Cottis and G. D. Kondylis, IEEE Trans. Antennas Propag. **43**, 154 (1995).
[30] W. T. Doyle, Phys. Rev. B **39**, 9852 (1989).
[31] B. Shanker and A. Lakhtakia, J. Phys. D: Appl. Phys. **26**, 1746 (1993).
[32] B. Shanker, J. Phys. D: Appl. Phys. **29**, 281 (1996).